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The Form of the Spectral Functions Associated with Sturm–Liouville Equations with Large Negative Potential

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1. INTRODUCTION

We consider the form of the spectral function, $\rho_\alpha(\lambda)$, associated with the linear, second order differential equation

$$y'' + (\lambda - q)y = 0 \qquad (0 \le x < \infty) \tag{1.1}$$

together with the initial condition

$$y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0. \tag{1.2}$$

We suppose that q is a real-valued member of $C^2[0,\infty)$, $q(x) \to -\infty$ as $x \to \infty$, and

$$\begin{gathered} \int_0^\infty (q')^2/|q|^{-5/2}\, dx < \infty, \\ \int_0^\infty |q''|\,|q|^{-3/2}\, dx < \infty, \\ \int^\infty |q|^{-1/2}\, dx \text{ is divergent}. \end{gathered} \tag{1.3}$$

It is known (see, for example, [3, pp. 255–256], [10, Sects. 5.7 and 5.10], and [7]) that under these circumstances ρ_α is absolutely continuous on $(-\infty,\infty)$. The problem of determining the form of ρ_α has received less attention. Asymptotic approximations were derived in [1], some of the earlier papers referenced therein, and, very recently, in [5]. Our object in

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this paper is to derive an exact representation for $\rho_\alpha(\lambda)$ in the form of a series valid for all $\lambda \geq \lambda_0$, where λ_0 is computable. We use this representation to derive an asymptotic expansion for $\rho_\alpha(\lambda)$ under circumstances somewhat more general than those considered in [5]. Much more general circumstances have been considered in [2], where one term asymptotics are obtained.

The basis for our analysis is a classical representation of ρ_α derived in [10] in terms of solutions of (1.1) with certain asymptotic properties. The solutions themselves are calculated using invariants of solutions of a Ricatti equation associated with (1.1). This is a method which was introduced in [9].

2. THE RESULTS

Let

$$I(t,\lambda) := \frac{q''}{4(\lambda - q)^{3/2}} + \frac{5(q')^2}{16(\lambda - q)^{5/2}}.$$

We suppose throughout that $\lambda - q(t) > 0$ and

$$\frac{I(t,\lambda)}{(\lambda - q(t))^{1/2}} \text{ is positive and decreasing for all } t \geq 0 \text{ and } \lambda \geq \lambda_0. \quad (2.1)$$

We note that (1.3) implies that

$$I(\cdot,\lambda) \in L^1[0,\infty) \quad \text{for all } \lambda \geq \lambda_0. \quad (2.2)$$

We set

$$\begin{aligned} v_2(x,\lambda) &:= (\lambda - q(x))^{1/2} \int_x^\infty \exp\left(2i\int_x^t (\lambda - q)^{1/2}\, ds\right) \\ &\qquad \times (\lambda - q(t))^{-1/2} I(t,\lambda)\, dt, \quad (2.3) \\ v_{j+1}(x,\lambda) &:= (\lambda - q(x))^{1/2} \int_x^\infty \exp\left(2i\int_x^t (\lambda - q)^{1/2}\, ds\right) \\ &\qquad \times \exp\left(2\sum_{l=2}^{j} \int_x^t v_l(x,\lambda)\, ds\right)(\lambda - q(t))^{-1/2} I(t,\lambda)\, dt \quad (2.4) \end{aligned}$$

for $j = 2, 3, \ldots$.

Let

$$
\begin{aligned}
v(x,\lambda) &:= i(\lambda-q)^{1/2} + \frac{q'}{4(\lambda-q)} + \sum_{j=2}^{\infty} v_j(x,\lambda),\\
S(x,\lambda) &:= \frac{q'}{4(\lambda-q)} + \operatorname{Re}\left\{\sum_{j=2}^{\infty} v_j(x,\lambda)\right\},\\
T(x,\lambda) &:= (\lambda-q)^{1/2} + \operatorname{Im}\left\{\sum_{j=2}^{\infty} v_j(x,\lambda)\right\}, \qquad (2.5)\\
S^*(x,\lambda) &:= \operatorname{Re}\left\{\sum_{j=2}^{\infty} v_j(x,\lambda)\right\},\\
T^*(x,\lambda) &:= \operatorname{Im}\left\{\sum_{j=2}^{\infty} v_j(x,\lambda)\right\}.
\end{aligned}
$$

THEOREM 1. *If $q \in C^2[0,\infty)$ satisfies* (1.3), (2.1), *and* (2.2), *then there exists $\lambda_0 \geq 0$ such that for all $\lambda \geq \lambda_0$,*

$$
\rho_0'(\lambda) = \frac{1}{\pi}(\lambda - q(0))^{1/2}\left[1 + \frac{T^*(0,\lambda)}{(\lambda-q(0))^{1/2}}\right]^2 \exp\left(-2\int_0^\infty S^*(t,\lambda)\,dt\right),
$$

$$
\rho_\alpha'(\lambda) = \frac{T(0,\lambda)^2 \exp\left(-2\int_0^\infty S^*(t,\lambda)\,dt\right)}{\pi(\lambda-q(0))^{1/2}\sin(\alpha)\left\{(S(0,\lambda)+\cot\alpha)^2 + T(0,\lambda)^2\right\}}
$$

for $\alpha \neq 0$.

In order to derive an asymptotic result from Theorem 1 we suppose that $q \in C^{N-1}[0,\infty)$ and $q^{(N-1)}$ is locally absolutely continuous for some $N \geq 4$ and we define

$$
a_2(x) := \frac{q'(x)}{4}, \qquad a_3(x) := \frac{iq''(x)}{8}, \qquad a_4(x) := \frac{-q^{(3)}(x)}{8},
$$

$$
a_{k+1}(x) := \frac{i}{2}\left\{a_k' + \frac{(k-2)}{2}q' a_{k-2} + \sum_{s=2}^{k-2} a_s a_{k-2}\right\}
$$

for $k = 2, \ldots, N-1$.

Let

$$\alpha(\lambda) := \max_{\substack{s+t=k \\ 2 \le s,t \le N \\ k=N,\dots,2N}} \int_0^\infty \frac{|a_s(x)|\,|a_t(x)|}{(\lambda - q(x))^{k/2}}\,dx, \tag{2.6}$$

$$\beta(\lambda) := \int_0^\infty |a_N'(x)|(\lambda - q(x))^{-N/2}\,dx. \tag{2.7}$$

THEOREM 2. *If, in addition to the hypotheses of Theorem* 1,

(i) $q^{(n)}(t)/(-q(t))^{(n+1)/2} \to 0$ *as* $t \to \infty$,

(ii) $q^{(n)}/(-q)^{(n+1)/2} \in L^2[0,\infty)$ *for* $n = 1, \dots, N-1$,

(iii) $q^{(N)}/(-q)^{N/2} \in L^1[0,\infty)$,

(iv) $(-1)^{m-1}\int_{x_1}^{x_2} q^{(2m-1)}(t)/(-q(t))^m\,dt \le C < \infty$ *for* $0 \le x_1 < x_2 < \infty$ *and* $m = 2,\dots,[N/2]$, *where* $[a]$ *denotes the integral part of* a, *then*

$$\left| T^*(x,\lambda) + i \sum_{\substack{k=3 \\ k \text{ odd}}}^{N} a_k(x)(\lambda - q(x))^{-k/2} \right| = O(\alpha(\lambda) + \beta(\lambda)),$$

$$\left| S^*(x,\lambda) - \sum_{\substack{k=4 \\ k \text{ even}}}^{N} a_k(x)(\lambda - q(x))^{-k/2} \right| = O(\alpha(\lambda) + \beta(\lambda))$$

as $\lambda \to \infty$.

3. A RESULT OF TITCHMARSH

Let $\phi(x,\lambda)$ denote the solution of (1.1) which satisfies the initial conditions

$$\phi(0,\lambda) = -\sin(\alpha), \qquad \phi'(0,\lambda) = \cos(\alpha). \tag{3.1}$$

It was shown by Titchmarsh [10, Sect. 5.10] that, under conditions (1.1) and (1.3), for $\lambda \ge \lambda_0$ there exist functions $a(\lambda)$ and $b(\lambda)$ with

$$\begin{aligned} \phi(x,\lambda) = {} & (\lambda - q(x))^{-1/4}\{a(\lambda)\cos(\xi(x)) + b(\lambda)\sin(\xi(x))\} \\ & + \Phi(x,\lambda), \end{aligned} \tag{3.2}$$

where $\Phi(x,\lambda) \to 0$ as $x \to \infty$ and $\xi(x) := \int_0^x (\lambda - q(t))^{1/2}\,dt$.

It was further shown that

$$\rho_\alpha'(\lambda) = \frac{1}{\pi\left(a(\lambda)^2 + b(\lambda)^2\right)}. \tag{3.3}$$

This result is not quite in Titchmarsh's form. For the modern notation we refer to [4] and [5]. It is our object to use the methods of [9] to derive a solution of (1.1) and (3.1) in the form of (3.2). This enables us to identify $a(\lambda)$ and $b(\lambda)$ which, by (3.3), give a representation of $\rho_\alpha'(\lambda)$.

4. THE METHOD OF SOLUTION

We give a brief account of the method which was introduced in [9] and the earlier papers cited therein.

Let $y(x, \lambda)$ denote a solution of (1.1) which is complex-valued in the sense that neither the real nor the imaginary part is identically zero. Since λ and q are real-valued, we may write

$$y = y_1 + iy_2, \tag{4.1}$$

where y_1 and y_2 are real-valued solutions of (1.1). We may also write

$$y(x, \lambda) := R(x, \lambda)e^{i\theta(x, \lambda)}, \tag{4.2}$$

where θ and R are real-valued. It follows from (4.2) that

$$\frac{y'}{y} = \frac{R'}{R} + i\theta'. \tag{4.3}$$

LEMMA 1. *If there exists $x_0 \in [0, \infty)$ with $R(x_0, \lambda)^2\theta'(x_0, \lambda) \neq 0$, then y_1 and y_2 are linearly independent solutions of* (1.1).

This lemma is proved in [9, Lemma 1].

LEMMA 2. *If $z(x, \lambda)$ is any real-valued solution of* (1.1) *and θ, R satisfy the condition of Lemma* 1, *then there exist real-valued p and ψ with $z(x, \lambda) \equiv p(x, \lambda)\cos(\psi(x, \lambda))$, where $p'/p = R'/R$ and $\psi' = \theta'$.*

This result is also proved in [9].

Lemma 2 provides a means to associate invariants with Eq. (1.1) in the following sense. Any complex-valued solution, y, gives rise to R'/R and θ' which, via Lemma 2, give a representation for all real-valued solutions.

Let $Y(x, \lambda)$ denote any complex-valued solution of (1.1) and define

$$S(x, \lambda) := \mathrm{Re}\left[\frac{Y'(x, \lambda)}{Y(x, \lambda)}\right], \qquad T(x, \lambda) := \mathrm{Im}\left[\frac{Y'(x, \lambda)}{Y(x, \lambda)}\right]. \tag{4.4}$$

If there exists x_0 with $T(x_0, \lambda) \neq 0$, then S and T are independent of the particular solution, Y, and any nontrivial, real-valued, solution z, may be expressed as

$$z(x, \lambda) = c_1 \exp\left(\int_0^x S(t, \lambda)\, dt\right) \cos\left(c_2 + \int_0^x T(t, \lambda)\, dt\right). \tag{4.5}$$

It was shown in [9] that c_1 and c_2 may be expressed in terms of the α of (3.1) as follows:

$$\text{if } \alpha = 0, \text{ then } c_2 = \frac{\pi}{2} \text{ and } c_1 = \frac{-1}{T(0, \lambda)}; \tag{4.6}$$

$$\begin{aligned} &\text{if } \alpha \neq 0, \text{ then } c_2 = \tan^{-1}\left(\frac{1}{T(0, \lambda)}(S(0, \lambda) + \cot(\alpha))\right), \\ &c_1 = \frac{-\sin(\alpha)}{\cos(c_2)}. \end{aligned} \tag{4.7}$$

It will be shown below that

$$S(x, \lambda) = \frac{q'}{4(\lambda - q)} + S^*(x, \lambda), \tag{4.8}$$

$$T(x, \lambda) = (\lambda - q)^{1/2} + T^*(x, \lambda). \tag{4.9}$$

Thus, any real-valued solution, z, of (1.1) may be written as

$$\begin{aligned} z(x, \lambda) = {} & \frac{1}{(\lambda - q(x))^{1/4}} \\ & \times \left\{ c_1 (\lambda - q(0))^{1/4} \exp\left(\int_0^x S^*(t, \lambda)\, dt\right) \right. \\ & \left. \times \cos\left[c_2 + \int_0^x (\lambda - q(t))^{1/2}\, dt + \int_0^x T^*(t, \lambda)\, dt \right] \right\}. \end{aligned} \tag{4.10}$$

If

$$\int_0^\infty S^*(t, \lambda)\, dt < \infty \quad \text{and} \quad \int_0^\infty T^*(t, \lambda)\, dt < \infty, \tag{4.11}$$

then

$$z(x, \lambda) = \frac{1}{(\lambda - q(x))^{1/4}} \left\{ c_1(\lambda - q(0))^{1/4} \exp\left(\int_0^\infty S^*(t, \lambda)\, dt \right) \right.$$
$$\times \cos\left(c_2 + \int_0^\infty T^*(t, \lambda)\, dt \right) \cos(\xi(x))$$
$$- c_1(\lambda - q(0))^{1/4} \exp\left(\int_0^\infty S^*(t, \lambda)\, dt \right)$$
$$\left. \times \sin\left(c_2 + \int_0^\infty T^*(t, \lambda)\, dt \right) \sin(\xi(x)) \right\}$$
$$+ \Phi(x, \lambda), \tag{4.12}$$

where $\Phi(x, \lambda) \to 0$ as $x \to \infty$ and $\xi(x) := \int_0^x (\lambda - q(t))^{1/2}\, dt$.

By comparing (3.2) and (4.12) we see that

$$a(\lambda) = c_1(\lambda - q(0))^{1/4} \exp\left(\int_0^\infty S^*(t, \lambda)\, dt \right) \cos\left(c_2 + \int_0^\infty T^*(t, \lambda)\, dt \right), \tag{4.13}$$

$$b(\lambda) = -c_1(\lambda - q(0))^{1/4} \exp\left(\int_0^\infty S^*(t, \lambda)\, dt \right) \sin\left(c_2 + \int_0^\infty T^*(t, \lambda)\, dt \right). \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$a(\lambda)^2 + b(\lambda)^2 = c_1^2(\lambda - q(0))^{1/2} \exp\left(2\int_0^\infty S^*(t, \lambda)\, dt \right). \tag{4.15}$$

By (3.3), (4.6), (4.7), and the fact that

$$T(0, \lambda) = (\lambda - q(0))^{1/2} + T^*(0, \lambda)$$
$$= (\lambda - q(0))^{1/2} \left\{ 1 + \frac{T^*(0, \lambda)}{(\lambda - q(0))^{1/2}} \right\},$$

we see that, for $\lambda \geq \lambda_0$,

$$\rho_0'(\lambda) = \frac{1}{\pi}(\lambda - q(0))^{1/2}\left[1 + \frac{T^*(0,\lambda)}{(\lambda - q(0))^{1/2}}\right]^2 \exp\left(-2\int_0^\infty S^*(t,\lambda)\,dt\right) \tag{4.16}$$

and, for $\alpha \neq 0$,

$$\rho_\alpha'(\lambda) = \frac{T(0,\lambda)^2 \exp\left(-2\int_0^\infty S^*(t,\lambda)\,dt\right)}{\pi(\lambda - q(0))^{1/2}\sin^2(\alpha)\left\{(S(0,\lambda) + \cot\alpha)^2 + T(0,\lambda)^2\right\}}. \tag{4.17}$$

These are the representations given in Theorem 1. We now show that S and T satisfy (2.4) and (2.5).

5. RICATTI EQUATIONS

Let $y(x,\lambda)$ denote a complex-valued solution of (1.1). The particular solution will be chosen below. It may readily be seen that if $v(x,\lambda) := y'(x,\lambda)/y(x,\lambda)$, then v satisfies the Ricatti equation

$$v' + v^2 + (\lambda - q) = 0. \tag{5.1}$$

We seek a solution of (5.1) in the form $v = \sum_{n=0}^\infty v_n$, substitution of which into (5.1) gives

$$v_0' + v_0^2 + (\lambda - q) + \sum_{n=1}^\infty v_n' + \sum_{n=1}^\infty v_n \sum_{m=0}^\infty v_m + v_0 \sum_{m=1}^\infty v_m = 0.$$

We choose

$$v_0(x,\lambda) = i(\lambda - q)^{1/2} \tag{5.2}$$

and then

$$v_0' + v_1' + 2v_0v_1 + v_1^2 + \sum_{n=2}^\infty v_n' + \sum_{n=2}^\infty v_n \sum_{m=0}^\infty v_m + \sum_{n=0}^{1} v_n \sum_{m=2}^\infty v_m = 0.$$

We choose v_1 so that $2v_0v_2 = -v_0'$, whence

$$v_1(x,\lambda) = \frac{q'}{4(\lambda - q)}. \tag{5.3}$$

Thus,

$$v_1' + v_1^2 + v_2' + 2(v_0 + v_1)v_2 + v_2^2 + \sum_{n=3}^{\infty} v_n' + \sum_{n=3}^{\infty} v_n \sum_{m=0}^{\infty} v_m + \sum_{n=1}^{2} v_n \sum_{m=3}^{\infty} v_m = 0. \tag{5.4}$$

We choose v_2 to satisfy

$$v_2' + 2(v_0 + v_1)v_2 = -(v_1' + v_1^2).$$

In particular,

$$v_2(x, \lambda) = \int_x^{\infty} \exp\left(2\int_x^t (v_0 + v_1)\, ds\right)(v_1' + v_1^2)\, dt.$$

It may be seen that

$$\exp\left(2\int_x^t v_1(s, \lambda)\, ds\right) = \frac{(\lambda - q(x))^{1/2}}{(\lambda - q(t))^{1/2}}$$

and

$$v_1' + v_1^2 = \frac{q''}{4(\lambda - q)} + \frac{5(q')^2}{16(\lambda - q)^2},$$

so

$$v_2(x, \lambda) = (\lambda - q(x))^{1/2} \int_x^{\infty} \exp\left(2i\int_x^t (\lambda - q)^{1/2}\, ds\right) I(t, \lambda)\, dt, \tag{5.5}$$

where

$$I(t, \lambda) := \frac{q'}{4(\lambda - q)^{3/2}} + \frac{5(q')^2}{16(\lambda - q)^{5/2}}.$$

Proceeding in this way we choose v_{J+1}, for $J \geq 1$, to satisfy

$$v_{J+1}' + 2\left(\sum_{k=0}^{J} v_k\right) v_{J+1} = -v_J^2 \tag{5.6}$$

and

$$v_{J+1}(x, \lambda) = (\lambda - q(x))^{1/2} \int_x^{\infty} \exp\left(2i\int_x^t (\lambda - q)^{1/2}\, ds\right) \times \exp\left(2\sum_{l=2}^{J} \int_x^t v_l\, ds\right)(\lambda - q(t))^{-1/2} v_J(t, \lambda)^2\, dt. \tag{5.7}$$

We recall from (2.1) and (2.2) that $(\lambda - q(t))^{-1/2} I(t,\lambda)$ is decreasing for all $t \geq 0$ and $\lambda \geq \lambda_0$ and $I(\cdot,\lambda) \in L^1[0,\infty)$ for all $\lambda \geq \lambda_0$.

LEMMA 3. *If λ_0 is so large that*

$$\lambda - q(x) > 0 \quad \text{for } x \geq 0$$

and

$$4\exp\left(8\int_0^\infty I(s,\lambda)\,ds\right)\int_0^\infty I(t,\lambda)\,dt < 1 \quad \text{for all } \lambda \geq \lambda_0,$$

then $|v_j(x,\lambda)| \leq (I(x,\lambda))/2^{j-3}$ *for* $x \geq 0$, $\lambda \geq \lambda_0$, *and* $j = 2, 3, \ldots$.

Proof. We use induction on j. Consider first the case $j = 2$,

$$\begin{aligned}
v_2(x,\lambda) &= (\lambda - q(x))^{1/2} \int_x^\infty \left\{\cos\left(2\int_x^t (\lambda - q)^{1/2}\,ds\right)\right. \\
&\qquad\qquad \left. + i\sin\left(2\int_x^t (\lambda - q)^{1/2}\,ds\right)\right\} I(t,\lambda)\,dt \\
&= (\lambda - q(x))^{1/2} \int_x^\infty \left\{2(\lambda - q(t))^{1/2} \cos\left(2\int_x^t (\lambda - q(s))^{1/2}\,ds\right)\right\} \\
&\quad \times \frac{I(t,\lambda)}{2(\lambda - q(t))^{1/2}}\,dt + i(\lambda - q(x))^{1/2} \\
&\quad \times \int_x^\infty \left\{2(\lambda - q(t))^{1/2} \sin\left(2\int_x^t (\lambda - q(s))^{1/2}\,ds\right)\right\} \\
&\quad \times \frac{I(t,\lambda)}{2(\lambda - q(t))^{1/2}}\,dt \\
&= \frac{I(x,\lambda)}{2} \int_{\xi_1(x)}^\infty \frac{d}{dt} \cos\left(\int_x^t (\lambda - q)^{1/2}\,ds\right) dt \\
&\quad + \frac{I(x,\lambda)}{2} \int_{\xi_2(x)}^\infty \frac{d}{dt} \sin\left(\int_x^t (\lambda - q)^{1/2}\,ds\right) dt
\end{aligned}$$

by the second mean value theorem, whence

$$|v_2(x,\lambda)| \leq 2I(x,\lambda).$$

Suppose now that the result were true for $j = 2, \dots, J$. We note that

$$\operatorname{Re}\left\{2\int_x^t \sum_{l=2}^{J} v_l(x, \lambda)\, ds\right\} \le 2\int_x^t \sum_{l=2}^{J} |v_l(s, \lambda)|\, ds$$

$$\le 2\left(\int_x^t I(s, \lambda)\, ds\right) \sum_{l=2}^{J} 2^{-(l-3)}$$

$$\le 4\left(\int_x^t I(s, \lambda)\, ds\right) \sum_{l=2}^{\infty} 2^{-(l-2)}$$

$$\le 8\int_x^t I(s, \lambda)\, ds \le 8\int_0^\infty I(s, \lambda)\, ds. \quad (5.8)$$

Now,

$$|v_{J+1}(x, \lambda)| \le (\lambda - q(x))^{1/2} \int_x^\infty \left|\exp\left(2i\int_x^t (\lambda - q)^{1/2}\, ds\right)\right|$$

$$\times \left|\exp\left(2\int_x^t \sum_{l=2}^{J} v_t\, ds\right)\right| (\lambda - q(t))^{-1/2} |v_J(t, \lambda)|^2\, dt$$

$$\le \frac{\exp\left(8\int_0^\infty I(s, \lambda)\, ds\right)}{2^{2J-6}} (\lambda - q(x))^{1/2}$$

$$\times \int_x^\infty (\lambda - q(t))^{-1/2} I(t, \lambda)^2\, dt$$

$$\le \frac{I(x, \lambda)}{2^{J-2}} \left(\frac{\exp\left(8\int_0^\infty I(s, \lambda)\, ds\right)}{2^{J-4}} \int_0^\infty I(t, \lambda)\, dt\right)$$

$$\le \frac{I(x, \lambda)}{2^{J-2}} \quad \text{by the second mean value theorem.}$$

The result now follows.

It may be seen from Lemma 3 that

$$v(x, \lambda) = i(\lambda - q)^{1/2} + \frac{q'}{4(\lambda - q)} + \sum_{j=2}^{\infty} v_j(x, \lambda),$$

where the right-hand side is uniformly absolutely convergent series of continuous functions for $\lambda \ge \lambda_0$, where λ_0 satisfies the condition of Lemma 3 in addition to (2.1) at (2.2). We also have, from (5.6), that for $j = 2, \dots,$

$$v_j' = -2\left(\sum_{l=0}^{j-1} v_l\right) v_j - v_{j-1}^2,$$

so, by Lemma 3 for $j \geq 3$,

$$|v_j'(x, \lambda)| \leq 2|v_j(x, \lambda)| \sum_{l=0}^{j-1} |v_l(x, \lambda)| + |v_{j-1}(x, \lambda)^2|$$

$$\leq 2\frac{I(x, \lambda)}{2^{j-3}}\left[|\lambda - q|^{1/2} + \left|\frac{q'}{4(\lambda - q)}\right| + 8I(x, \lambda)\right] + \frac{I(x, \lambda)^2}{2^{2j-6}}$$

$$\leq \frac{I(x, \lambda)}{2^{j-4}}\left\{|\lambda - q|^{1/2} + \left|\frac{q'}{4(\lambda - q)}\right| + 8I(x, \lambda) + \frac{I(x, \lambda)}{2^{j-2}}\right\}.$$

So $\sum_{j=0}^{\infty} v_j'(x, \lambda)$ is uniformly, absolutely convergent for $x \geq 0$ and $\lambda \geq \lambda_0$.

We have shown that

$$v(x, \lambda) = i(\lambda - q)^{1/2} + \frac{q'}{4(\lambda - q)} + \sum_{j=2}^{\infty} v_j(x, \lambda)$$

is a complex-valued solution of (5.1) and, in the notation of (4.4),

$$T(x, \lambda) = (\lambda - q(x))^{1/2} + \sum_{j=2}^{\infty} \operatorname{Im}\{v_j(x, \lambda)\},$$

$$S(x, \lambda) = \frac{q'(x)}{4(\lambda - q(x))} + \sum_{j=2}^{\infty} \operatorname{Re}\{v_j(x, \lambda)\}.$$

We note from Lemma 3 there exists λ_0 such that for $\lambda \geq \lambda_0$ there is an x_0 such that $\operatorname{Im}\{v(x_0, \lambda)\} \neq 0$. Theorem 1 now follows from (4.16) and (4.17).

6. A METHOD OF APPROXIMATION

In order to prove Theorem 2 we require a means of approximating asymptotically the solution, v, of the Ricatti equation (5.1).

Let $u(x, \lambda)$ denote a solution of

$$u' + u^2 + \lambda - q = B(t, \lambda), \tag{6.1}$$

where B is a L^1 function to be chosen below. We suppose that

$$\lim_{x \to \infty} (v(x, \lambda) - u(x, \lambda)) = 0. \tag{6.2}$$

Subtraction of (6.1) from (5.1) yields

$$(v - u)' + (v + u)(v - u) = -B, \tag{6.3}$$

whence, by (6.2),

$$v(x,\lambda) - u(x,\lambda) = \int_x^\infty \exp\left(\int_x^t (v+u)\,ds\right) B(t,\lambda)\,dt. \tag{6.4}$$

Our method for approximating the solution of (5.1) is to choose a function $u(x, \lambda)$ and hence a $B(x, \lambda)$ which satisfies (6.2) and makes the right-hand side of (6.4) small for large λ.

LEMMA 4. *If* $\mathrm{Re}\{\int_{x_1}^{x_2} (v(s,\lambda) - u(x,\lambda))\,ds\} \le C_1$ *for* $0 \le x_1 \le x_2 < \infty$ *for all* $\lambda \ge \lambda_0$, *then* $|v(x,\lambda) - u(x,\lambda)| \le C \int_x^\infty |B(t,\lambda)|\,dt$ *for* $\lambda \ge \lambda_0$ *and* $x \ge 0$.

Proof. The proof is immediate from (6.4).

7. PROOF OF THEOREM 2

We now choose a function, $u(x, \lambda)$, satisfying (6.2) and a corresponding function $B(x, \lambda)$.

We write

$$u(x,\lambda) = i(\lambda - q)^{1/2} + \sum_{k=2}^{N} a_k(x)(\lambda - q(x))^{-k/2}. \tag{7.1}$$

Substitution of (7.1) into the left-hand side of (6.1) and selection of the $a_k(x)$ so that the coefficient of $(\lambda - q(x))^{-k/2}$ is identically zero for $k = 0, \ldots, N-1$ leads, in a similar way to that used in [9, (6.1)], to the choice

$$\begin{gathered} a_2 = \frac{q'}{4}, \qquad a_3 = \frac{iq''}{8}, \qquad a_4 = \frac{-q^{(3)}}{16}, \\ a_{k+1} = \frac{i}{2}\left\{a_k' + \frac{k-2}{2} q' a_{k-2} + \sum_{s=2}^{k-2} a_s a_{k-2}\right\} \end{gathered} \tag{7.2}$$

for $k = 4, \ldots, N-1$. We then have that

$$\begin{aligned} B(x,\lambda) = a_N'(\lambda - q)^{-N/2} &+ \sum_{k=N}^{N+2} \frac{k-2}{2} q' a_{k-2}(\lambda - q)^{-k/2} \\ &+ \sum_{k=N}^{2N} (\lambda - q)^{-k/2} \sum_{s+t=k} a_s a_t. \end{aligned} \tag{7.3}$$

LEMMA 5. *$a_k(x)$ is real-valued if k is even and purely imaginary if k is odd.*

Proof. The proof follows readily from (7.2) by induction.

LEMMA 6. *If $q \in C^{N-1}[0,\infty)$, $q^{(N-1)}$ is locally absolutely continuous and*

(i) $q^{(k)}/(-q)^{(k+1)/2} \in L^2[0,\infty)$,

(ii) $q^{(k)}(t)/(-q(t))^{(k+1)/2} \to 0$ *as* $t \to \infty$

for $k = 1, \dots, N-1$, *then*

$$\frac{a_j^{(k)}(t)}{(-q(t))^{(j+k)/2}} = \frac{i^{j-2}q^{(j+k-1)}(t)}{2^j(-q(t))^{(k+j)/2}} + \frac{l}{(-q(t))^{1/2}}$$

for $k = 0, N-j+1$, $j = 2, \dots, N$, *and* $t \geq 0$, *where* $l(\cdot)$ *denotes an* L^1 *function with* $l(t) \to 0$ *as* $t \to \infty$.

Proof. We prove the result by induction. It is trivial for $j = 2$, 3, and 4. Suppose the result were true for some $J \geq 4$. By Leibnitz' rule and (7.2),

$$\begin{aligned} \frac{a_{J+1}^{(k)}}{(-q)^{(J+k+1)/2}} &= \frac{ia_J^{(k+1)}}{2(-q)^{(J+k+1)/2}} \\ &\quad + \frac{i(J-2)}{4(-q)^{1/2}} \sum_{l=0}^{k} \binom{k}{l} \frac{q^{(k-l+1)}}{(-q)^{(k-l+2)/2}} \frac{a_{J-2}^{(l)}}{(-q)^{(J+l-2)/2}} \\ &\quad + \frac{i}{2(-q)^{1/2}} \sum_{s=2}^{J-2} \sum_{l=0}^{k} \binom{k}{l} \frac{a_s^{(k-l)}}{(-q)^{(k-l+s)/2}} \frac{a_{J-s}^{(l)}}{(-q)^{(J-s+l)/2}}. \end{aligned}$$

The result now follows from the induction hypothesis by the Schwarz inequality.

We see from Lemmas 3, 5, and 6 and condition (iv) of Theorem 2, that

$$\mathrm{Re}\left\{\int_{x_1}^{x_2} (v(s,\lambda) + u(s,\lambda))\, ds\right\} \leq C_1 \quad \text{for } 0 \leq x_1 \leq x_2 < \infty$$

and $\lambda \geq \lambda_0$. It also follows from Lemmas 3 and 6 together with condition (i) of Theorem 2 that $\lim_{x\to\infty}(u(x,\lambda) - v(x,\lambda)) = 0$. We thus have from Lemma 4 that

$$|u(x,\lambda) - v(x,\lambda)| \leq C\int_x^\infty |B(t,\lambda)|\, dt \quad \text{for } x \geq 0,$$

where B is given by (7.3). The result now follows from (2.6) and (2.7).

8. A RESULT OF EASTHAM

In [5], Eastham derives an expression for the asymptotic form of $\rho_\alpha(\lambda)$ when q is $M + 2$ times differentiable

$$q(x) \leq -kx^c \quad \text{for } 0 < c \leq 2,\ k > 0, \tag{8.1}$$

and

$$|q^{(v)}(x)| \leq (\text{const})\, x^{c-v} \quad \text{for } 1 \leq v \leq M + 2. \tag{8.2}$$

It may readily be seen that under these circumstances

$$\left| \frac{q^{(n)}(x)}{(-q(x))^{(n+1)/2}} \right| \leq (\text{const.})\, x^{(c/2)(1-n)-n} \leq \text{const.}\, x^{-1}$$

for $n \geq 1$ and

$$\left| \frac{q^{(2m-1)}}{(-q)^m} \right| \leq (\text{const.})\, x^{c(1-m)-2m+1} \leq x^{-3} \quad \text{for } m \geq 2,$$

so that Theorem 2 is applicable. The computation of the error term is more complicated, but is may be shown that for each $n \geq 2$, $a_n(x)$ is bounded by (const.)$x^{[(n+1)/3](c-1)-\Gamma n}$, where $[(n + 1)/3]$ denotes the integer part of $(n + 1)/3$ and Γ_n is the remainder when $n + 1$ is divided by 3. The α and β functions of (2.6) and (2.7) may then be estimated using [5, Lemma 3].

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